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# Compatible, Energy and Symmetry Preserving 2D Lagrangian Hydrodynamics in $rz$ - Cylindrical Coordinates

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## Abstract

We present a new discretization for 2D Lagrangian hydrodynamics in  $rz$  geometry (cylindrical coordinates), which is total energy conserving and symmetry preserving.

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*Keywords:* Lagrangian hydrodynamics,, 2D axisymmetric  $rz$ , spatial symmetry, conservation

## 1. Introduction

We start with brief review of gas dynamics equations in axisymmetric geometry -  $rz$  coordinates.

*Momentum Equation.* The general form of momentum equation for a Lagrangian fluid parcel  $V(t)$  is

$$\frac{d}{dt} \left( \int_{V(t)} \mathbf{u} \rho dV \right) = - \oint_{\partial V(t)} p \mathbf{n} dS. \quad (1)$$

If the general formula (1) is applied to the control volume presented in Fig. 1, one can obtain the following conservative form of the momentum equation in  $r - z$  axisymmetric geometry

$$r \rho \frac{du}{dt} = - \left( \frac{\partial(r p)}{\partial r} - p \right), \quad r \rho \frac{dv}{dt} = - \frac{\partial(r p)}{\partial z} \quad (2)$$

Here  $u$  and  $v$  are the  $r$  and  $z$  components of the velocity vector.

The equation for specific internal energy,  $\varepsilon$ , has the following form

$$r \rho \frac{d\varepsilon}{dt} = -p \left( \frac{\partial(r u)}{\partial r} + \frac{\partial(r v)}{\partial z} \right). \quad (3)$$

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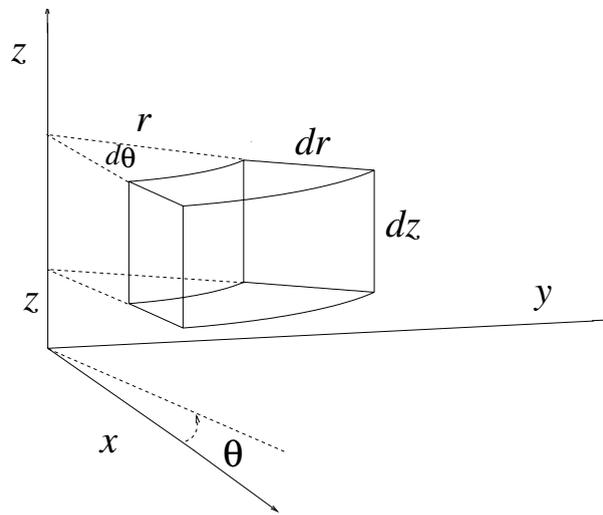


Figure 1: Control volume

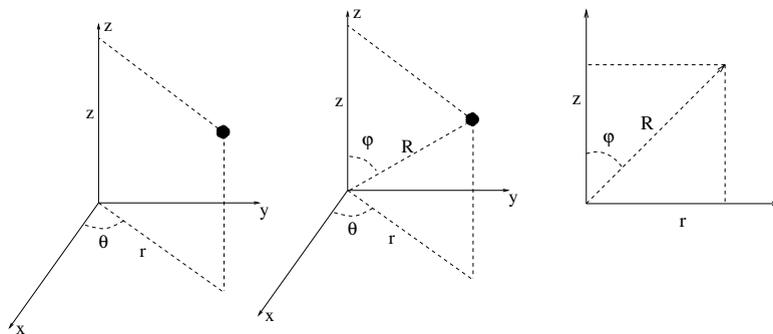


Figure 2: Cartesian -  $(x, y, z)$ , cylindrical -  $(r = \sqrt{x^2 + y^2}, z, \theta)$ , and spherical Coordinates -  $(R = \sqrt{r^2 + z^2}, \theta, \varphi)$

To compute complex flows with shocks it is very important to conserve total energy, which is defined as follows. The specific total energy,  $E$ , for axisymmetric flow is

$$E = \int_{V(t)} \left( \varepsilon + \frac{u^2 + v^2}{2} \right) \rho dV. \tag{4}$$

The conservation law for total energy is

$$\frac{d}{dt} \left( \int_{V(t)} E \rho dV \right) = - \oint_{\partial V(t)} p \mathbf{u} \cdot \mathbf{n} dS. \tag{5}$$

If initial and boundary conditions are specified appropriately, then the equations in cylindrical coordinates allow a spherically symmetric solution. That is, density, internal energy and pressure depend only upon the spherical radius  $(R = \sqrt{r^2 + z^2}, \rho(r, z) = \rho(R, t), \varepsilon(r, z) = \varepsilon(R, t), p(r, z) = p(R))$ , and velocity can be represented as  $u = U(R) \sin \varphi, v = U(R) \cos \varphi$  for coordinate systems represented in Fig. (2).

Boundary conditions for spherically symmetric problems are presented in Fig. 3.

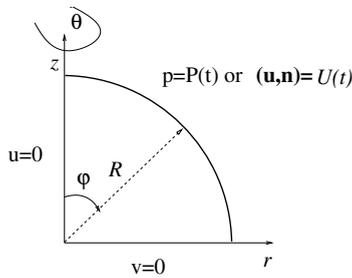


Figure 3: Boundary conditions for spherically symmetric problem

For spherically symmetric problems, acceleration  $\mathbf{a} = (a_u, a_v)$  is directed in the radial direction and its amplitude depends only upon the spherical radius  $R$ :

$$a_u = -\frac{1}{\rho(R)} \frac{\partial p(R)}{\partial R} \sin \varphi, \quad a_v = -\frac{1}{\rho(R)} \frac{\partial p(R)}{\partial R} \cos \varphi. \quad (6)$$

The equation for internal energy has the following form:

$$\frac{d\varepsilon}{dt} = -\frac{1}{\rho(R)} p(R) \cdot \left( \frac{1}{R^2} \frac{\partial(R^2 U)}{\partial R} \right), \quad (7)$$

that is  $\varepsilon = \varepsilon(R, t)$ .

In this paper we present a new staggered discretization for 2D Lagrangian hydrodynamics in  $rz$  geometry on logically rectangular grid, which is compatible with described partial differential equations, that is, total energy conserving and symmetry preserving.

We want to mention recent paper [3] devoted to cell-centered discretization of 2D Lagrangian hydrodynamics in  $rz$  geometry.

We show that our discretization preserves spherical symmetry on polar equiangular meshes. It has a consistent definition of kinetic energy in the zone that is exact for a constant velocity field.

The method is based on ideas presented in [2, 1], where the authors use a special procedure to distribute zonal mass to corners of the zone (subzonal masses). The momentum equation is discretized in its "Cartesian" form with a special definition of "planar" masses (area-weighted).

Two principal contributions of this paper are as follows: first is a definition of "planar" subzonal mass for nodes on the  $z$  axis ( $r = 0$ ) that does not require a special procedure for movement of these nodes. Second is proof that the discretization preserves spherical symmetry including analysis internal energy equation.

We present numerical examples that demonstrate the robustness of the new method.

## 2. Compatible staggered discretization

For simplicity of presentation, we will consider semi-discrete discretizations; that is we will keep time continuous and consider only spatial discretization.

In this paper we consider logically rectangular grid, Fig. 4, where each point,  $p$ , can be enumerated by two integer indices  $(i, j)$ , and each zone,  $z$ , enumerated by two half indices  $i + \frac{1}{2}, j + \frac{1}{2}$ .

In the staggered discretization point quantities are coordinates,  $(r_p, z_p)$ , and velocity components,  $(u_p, v_p)$ . Zonal quantities are volume,  $V_z$ , density,  $\rho_z$ , mass  $m_z$ , internal energy,  $\varepsilon_z$  and pressure  $p_z$ .

For Lagrangian methods:  $d(m_z)/dt = 0$  with  $m_z = \rho_z|_{t=0} \cdot V_z|_{t=0}$ . Therefore, for any time moment,  $t$  density can be defined as follows  $\rho_z(t) = m_z/V_z(t)$ , which plays the role of discrete continuity equation. Also for Lagrangian methods each point moves with fluid velocity, that is,  $d(r_p)/dt = u_p, d(z_p)/dt = v_p$ .

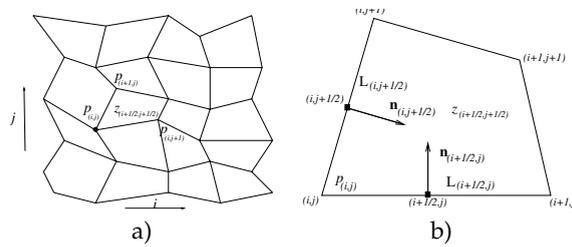


Figure 4: Logically rectangular grid: a) Entire grid, b) Zone  $(i + \frac{1}{2}, j + \frac{1}{2})$  and corresponding notations

Zonal internal energy is  $\mathcal{E}_p = m_z \varepsilon_z$ . Point momentum,  $\mu_p$ , and kinetic energy,  $K_p$ , are defined as follows

$$\mu_p = m_p \mathbf{u}_p, \quad K_p = m_p \frac{|\mathbf{u}_p|^2}{2}$$

where point mass,  $m_p$ , remains to be defined.

There are following obvious requirements for point mass.

$$d(m_p)/dt = 0; \quad \sum_z m_z = \sum_p m_p.$$

Following [2], we introduce additional Lagrangian objects, so-called, subzonal masses,  $m_z^p$ , such that  $d(m_z^p)/dt = 0$  and

$$\sum_{p \in \mathcal{P}(z)} m_z^p = m_z, \tag{8}$$

where  $\mathcal{P}(z)$  is set of vertices of zone  $z$ . Then point mass can be defined as follows  $m_p = \sum_{z \in \mathcal{Z}(p)} m_z^p$  where  $\mathcal{Z}(p)$  is set of zones which have point  $p$  as vertex.

It is important to note that if one defines subzonal volume  $V_z^p(t)$  in some consistent way, such that  $\sum_{p \in \mathcal{P}(z)} V_z^p(t) = V_z(t)$ , then it leads to natural definition of subzonal density

$$\rho_z^p(t) = m_z^p / V_z^p(t). \tag{9}$$

The generic compatible form of the discrete momentum and internal energy equations is

$$m_p \frac{d\mathbf{u}_p}{dt} = \sum_{z \in \mathcal{P}(z)} \mathbf{f}_z^p, \quad m_z \frac{d\varepsilon_z}{dt} = - \sum_{p \in \mathcal{Z}(p)} \mathbf{f}_z^p \cdot \mathbf{u}_p.$$

Therefore the spatial discretization is completely defined if we define subzonal masses  $m_z^p$  and subzonal forces  $\mathbf{f}_z^p$ . In simplest case, which we will consider in this paper, subzonal force depends on the geometry of the zone and zonal pressure.

For compatible discretizations, evolution of the total kinetic energy,  $K = \sum_p K_p$ , is  $\frac{dK}{dt} = \sum_p \left( \sum_{z \in \mathcal{P}(z)} \mathbf{f}_z^p \cdot \mathbf{u}_p \right)$ , and evolution of total internal energy,  $\mathcal{E} = \sum_p \mathcal{E}_p$ , is

$\frac{d\mathcal{E}}{dt} = - \sum_z \left( \sum_{p \in \mathcal{Z}(p)} \mathbf{f}_z^p \cdot \mathbf{u}_p \right)$ . Therefore the change in total energy,  $E = \mathcal{K} + \mathcal{E}$ , is

$$\frac{dE}{dt} = \sum_p \left( \sum_{z \in \mathcal{P}(z)} \mathbf{f}_z^p \cdot \mathbf{u}_p \right) - \sum_z \left( \sum_{p \in \mathcal{Z}(p)} \mathbf{f}_z^p \cdot \mathbf{u}_p \right) = \text{boundary terms},$$

which is the discrete form of total energy conservation.

### 3. Definition of subzonal masses

The discretization of momentum equation that preserves spherical symmetry is based on the non-conservative (“Cartesian”) form of the momentum equation

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial r}, \quad \rho \frac{dv}{dt} = -\frac{\partial p}{\partial z}, \tag{10}$$

This is equivalent to (2) in the differential case.

A control volume derivation of the discrete “Cartesian” form of the momentum equation gives

$$\langle \rho A \rangle_p(t) \frac{d\mathbf{u}_p}{dt} = \sum_{z \in \mathcal{Z}(p)} (\mathbf{f}_{cart})_z^p, \tag{11}$$

where  $\langle \rho A \rangle_p(t) \approx \int_{z(t)} \rho(r, z, t) dr dz$ . The discrete form of  $\langle \rho A \rangle_p(t)$  is  $\langle \rho A \rangle_p(t) = \sum_{z \in \mathcal{Z}(p)} (\rho_z^p)(t) (A_z^p)(t)$ , where  $(\rho_z^p)(t)$  and  $(A_z^p)(t)$  remain to be defined. For symmetry preservation it can be any symmetric quadrature around the point. The necessary condition for zero order approximation is

$$\sum_{p \in \mathcal{P}(z)} (A_z^p)(t) = (A_z)(t). \tag{12}$$

The discrete equation (11) is not in compatible form, therefore to construct a conservative discretization we need to define  $(\rho_z^p)(t)$  and  $(A_z^p)(t)$  in such a way that equation (11) can be rewritten in equivalent compatible form:

$$m_p \frac{d\mathbf{u}_p}{dt} = \sum_{z \in \mathcal{P}(z)} \mathbf{f}_z^p, \tag{13}$$

where  $m_p$  corresponds to true cylindrical mass,  $\int_{V(p)} \rho r dr dz$ . Comparison of (13) and (11) leads to following definition of  $m_p$  and  $\mathbf{f}_z^p$

$$m_p = r_p(t) \left( \sum_{z \in \mathcal{Z}(p)} (\rho_z^p)(t) (A_z^p)(t) \right), \quad \mathbf{f}_z^p(t) = r_p(t) (\mathbf{f}_{cart})_z^p(t). \tag{14}$$

Using the notion of subzonal masses, we construct point point mass as follows

$$m_p = \sum_{z \in \mathcal{Z}(p)} m_z^p, \quad m_z^p = (\rho_z^p)(t) r_p(t) (A_z^p)(t) = (\rho_z^p)|_{t=0} r_p|_{t=0} (A_z^p)|_{t=0}. \tag{15}$$

This leads to a natural definition of subzonal volume

$$V_z^p(t) = r_p(t) A_z^p(t). \tag{16}$$

Equation (8) leads to the following requirement for  $A_z^p$

$$\sum_{p \in \mathcal{P}(z)} (V_z^p)(t) = \sum_{p \in \mathcal{P}(z)} (r_p)(t) (A_z^p)(t) = V_z(t). \tag{17}$$

Therefore, we need to find  $A_z^p(t)$  that satisfies two conditions (12) and (17). One of the possible solutions for quad mesh as presented in [2] is:

$$A_z^1 = \frac{5A_{41} + 5A_{12} + A_{23} + A_{34}}{12}, \quad A_z^2 = \frac{A_{41} + 5A_{12} + 5A_{23} + A_{34}}{12},$$

$$A_z^3 = \frac{A_{41} + A_{12} + 5A_{23} + 5A_{34}}{12}, \quad A_z^4 = \frac{5A_{41} + A_{12} + A_{23} + 5A_{34}}{12},$$

where corresponding triangles are shown in Fig. 5

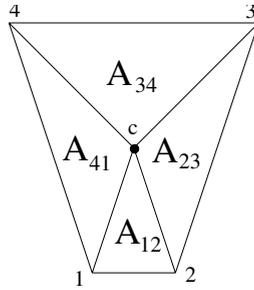


Figure 5: Volume Distribution. Geometric center of the quad is  $r_c = (r_1 + r_2 + r_3 + r_4)/4$ ,  $z_c = (z_1 + z_2 + z_3 + z_4)/4$

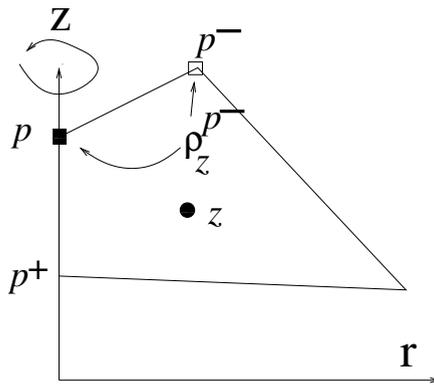


Figure 6: Definition of subzonal density for points on z axis

For points that are not on z axis, the subzonal volume  $V_z^p$  defined by equation (16) is positive and therefore the subzonal density can be defined using equation (9). However, for points on the z axis,  $m_z^p = 0$  and  $V_z^p = 0$ . Therefore we need to define the corresponding subzonal density using some other means. The definition should satisfy some minimal requirements of consistency and also preserve spherical symmetry on special meshes.

For a logically rectangular mesh, each point on the z axis is connected by a zone edge to only one point that is not on the axis. For example, in Fig. 6 for point  $p$  on z axis such point is  $p^-$ . In this situation we set

$$\rho_z^p(t) = \rho_z^{p^-}(t). \tag{18}$$

It should be noted that such a definition of subzonal density does not affect the corresponding subzonal mass which remains zero because the corresponding subzonal volume is zero.

Now let us consider how this modification affects the logic of the compatible discretization. First, if we define zonal kinetic energy as follows

$$K_z = \sum_{p \in \mathcal{P}(z)} \frac{m_z^p}{2} |\mathbf{u}_p|^2 \sim \int_{V_z} \rho \frac{|\mathbf{u}_p|^2}{2} dV$$

then the minimal consistency condition means that if  $\mathbf{u}$  is constant then  $K_z$  should be exact. This is clearly true in our discretization because of the definition of subzonal volumes.

Second, kinetic energy does not depend on the velocity of points on the  $z$  axis because the corresponding  $m_z^p = 0$ . Therefore, movement of points on the boundary (which is affected by the definition of corresponding subzonal masses) does not contribute to instantaneous energy balance. Third, the definition of zonal internal energy,  $\mathcal{E}_z = m_z \varepsilon_z$ , does not depend on  $m_z^p$  as long as the corresponding  $m_z^p$  sum to  $m_z$ , which is true in our case. Therefore the logic of compatible discretization is not affected and our discretization conserves total energy.

#### 4. Discrete equations on logically rectangular grid

On logically rectangular grids, the discretization of "Cartesian" forces follows from the definition of the "Cartesian" gradient operator

$$\int_{A(t)} \mathbf{grad}_{cart} p \, dA = \oint_{\partial A(t)} p \, \mathbf{n} \, dl;$$

Using notations for points and zones for such grids, we obtain the following expression for the subzonal forces

$$(\mathbf{f}_{cart})_{i+\frac{1}{2},j+\frac{1}{2}}^{i,j} = p_{i+\frac{1}{2},j+\frac{1}{2}} \left( \frac{L_{i+\frac{1}{2},j}}{2} \mathbf{n}_{i+\frac{1}{2},j} + \frac{L_{i,j+\frac{1}{2}}}{2} \mathbf{n}_{i,j+\frac{1}{2}} \right).$$

Similar discretization of momentum equation yields

$$\begin{aligned} \langle \rho A \rangle_{i,j}^n \frac{d\mathbf{u}_{i,j}}{dt} = & \\ -\frac{1}{2} \{ & [L_{i+\frac{1}{2},j} (p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i+\frac{1}{2},j-\frac{1}{2}}) \mathbf{n}_{i+\frac{1}{2},j} + L_{i-\frac{1}{2},j} (p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) \mathbf{n}_{i-\frac{1}{2},j}] + \\ & [L_{i,j+\frac{1}{2}} (p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}}) \mathbf{n}_{i,j+\frac{1}{2}} + L_{i,j-\frac{1}{2}} (p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) \mathbf{n}_{i,j-\frac{1}{2}}] \}. \end{aligned}$$

Equation for internal energy looks as follows

$$\begin{aligned} m_{i+\frac{1}{2},j+\frac{1}{2}} \frac{\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - \varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}^n}{\Delta t} = & -p_{i+\frac{1}{2},j+\frac{1}{2}} \left\{ \right. \\ & \left[ L_{i+\frac{1}{2},j+1} \mathbf{n}_{i+\frac{1}{2},j+1} \cdot \frac{r_{i,j+1}^n \mathbf{u}_{i,j+1} + r_{i+1,j+1}^n \mathbf{u}_{i+1,j+1}}{2} - L_{i+\frac{1}{2},j} \mathbf{n}_{i+\frac{1}{2},j} \cdot \frac{r_{i,j}^n \mathbf{u}_{i,j} + r_{i+1,j}^n \mathbf{u}_{i+1,j}}{2} \right] + \\ & \left. \left[ L_{i+1,j+\frac{1}{2}} \mathbf{n}_{i+1,j+\frac{1}{2}} \cdot \frac{r_{i+1,j}^n \mathbf{u}_{i+1,j} + r_{i+1,j+1}^n \mathbf{u}_{i+1,j+1}}{2} - L_{i,j+\frac{1}{2}} \mathbf{n}_{i,j+\frac{1}{2}} \cdot \frac{r_{i,j}^n \mathbf{u}_{i,j} + r_{i,j+1}^n \mathbf{u}_{i,j+1}}{2} \right] \right\} \end{aligned}$$

#### 5. Preservation of spherical symmetry on an equiangular polar mesh

In this section we show that our new discretization preserves spherical symmetry on special equiangular polar meshes, Fig. 7 a).

Coordinates of points of such a mesh are defined as follows

$$\begin{aligned} r_{i,j} &= R_j \sin \varphi_i, \quad z_{i,j} = R_j \cos \varphi_i; \quad i = 1, \dots, I; j = 1, \dots, J, \\ \varphi_i &= (i-1) \Delta \varphi; \quad \Delta \varphi = \frac{\pi/2}{I-1}. \end{aligned}$$

The statement of exact preservation of spherical symmetry on polar mesh can be formulated as follows. Corresponding components of the velocity are equal to zero on axis:  $u_{1,j}(t) = 0; v_{I,j}(t) = 0$ . Both components of the velocity are zero at the origin:  $u_{1,1}(t) = v_{1,1}(t) = 0$ . On the outer boundary, one can specify pressure as the function of the time  $p_{i+1/2,J}(t) = P_{top}(t)$ , or the normal component of the velocity as function of time  $u_{i,J}(t) \sin \theta_i + v_{i,J}(t) \cos \theta_i = U(t^n)$ .

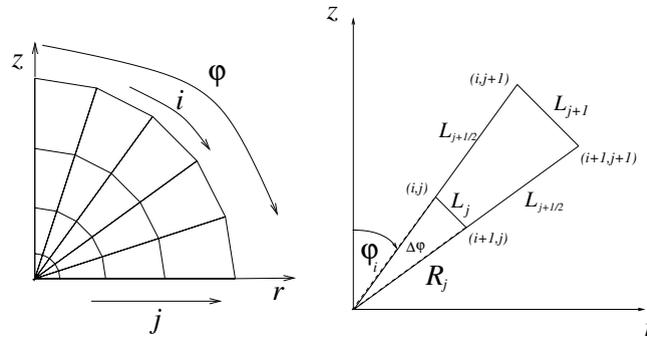


Figure 7: Equiangular polar mesh: a) Entire mesh, b) One cell

Initial conditions for scalar functions are as follows:  $\rho_{i+1/2,j+1/2}|_{t=0} = \rho_{j+1/2}|_{t=0}$ ;  $p_{i+1/2,j+1/2}|_{t=0} = p_{j+1/2}|_{t=0}$ ,  $\varepsilon_{i+1/2,j+1/2}|_{t=0} = \varepsilon_{j+1/2}|_{t=0}$ . Initial velocity is directed radially and its magnitude depends only on  $j$ :  $u_{i,j}|_{t=0} = U_j|_{t=0} \cdot \sin \theta_i$  and  $v_{i,j}|_{t=0} = U_j|_{t=0} \cdot \cos \theta_i$ .

If for these initial and boundary conditions, the density, internal energy and pressure depend only on  $j$  and the velocity is spherical  $u_{i,j}(t) = U_j(t) \sin \theta_i$  and  $v_{i,j}(t) = U_j(t) \cos \theta_i$  at all later times, then we say that the finite difference scheme preserves spherical symmetry.

On a polar equiangular mesh, the momentum equation can be written in the following form

$$\mathbf{a}_{i,j} = \frac{d\mathbf{u}_{i,j}}{dt} = \left\{ \frac{R_j (p_{j+\frac{1}{2}} - p_{j-\frac{1}{2}})}{\left[ \left[ \rho_{j+\frac{1}{2}} (R_{j+1} - R_j) \frac{2R_j+R_{j+1}}{3} \right] + \left[ \rho_{j-\frac{1}{2}} (R_j - R_{j-1}) \frac{2R_j+R_{j-1}}{3} \right] \right] / 2} \right\} \mathbf{e}_i^R,$$

where  $\mathbf{e}_i^R = (\sin \varphi_i, \cos \varphi_i)^T$ . Therefore, in the discrete case, acceleration is directed in the radial direction, and its magnitude depends only on  $j$ , that is, the momentum equation preserves spherical symmetry.

Similarly, the equation for specific internal energy also depends only on radius

$$\frac{d\varepsilon_{i+\frac{1}{2},j+\frac{1}{2}}}{dt} = \frac{1}{\rho_{j+\frac{1}{2}}} p_{j+\frac{1}{2}} \frac{1}{\left[ (R_j)^2 + R_{j+1}R_j + (R_{j+1})^2 \right] / 3} \frac{(R_{j+1})^2 U_{j+1} - (R_j)^2 U_j}{R_{j+1} - R_j}.$$

To numerically demonstrate the preservation of spherical symmetry, we present results for the so-called Noh’s spherical problem calculated in cylindrical geometry [4]. This problem has been used extensively to illustrate the difficulties of preserving spherical symmetry in cylindrical geometry. Initially the velocity is directed radially inward with a magnitude of 1.0, the density is unity, and the internal energy is zero. In Fig. 8 we present computational mesh at  $t = 0.6$  and density. The numerical results clearly confirm our theory; that is, the numerical solution preserves spherical symmetry exactly.

### 6. Conclusion

We want to emphasize that methodology for construction compatible total energy conserving staggered discretization for Lagrangian hydro in 2D  $rz$  coordinates described in this paper can be used for general polygonal meshes. This generalization will be described in future paper. We also plan to perform more tests especially on polygonal meshes to demonstrate robustness of our new method and compare our method with other methods, for example, methods presented in [3].

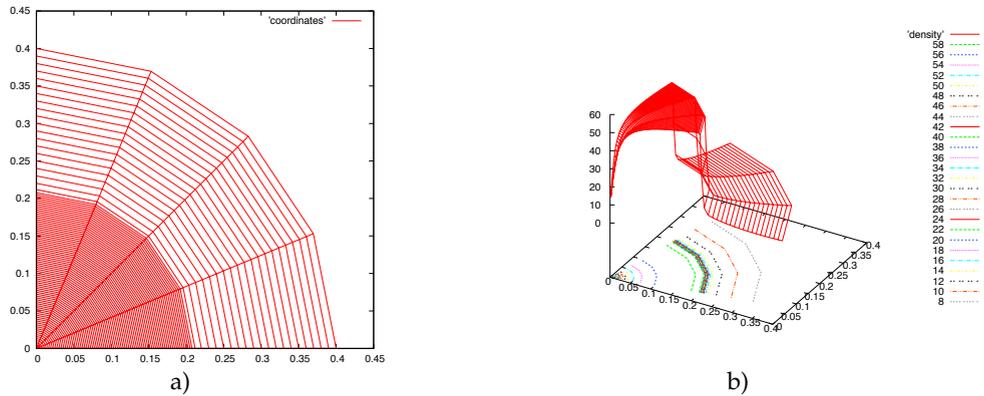


Figure 8: Noh Problem: a) Mesh, b) Density

## 7. Acknowledgements

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## References

- [1] A. J. Barlow, *A compatible finite element multi-material ALE hydrodynamics algorithm*, Int. J. Numer. Meth. Fluids vol:56, pp. 953-964 (2008).
- [2] E. J. Caramana, D. E. Burton, M. J. Shashkov and P. P. Whalen. *The construction of compatible hydrodynamics algorithms utilizing conservation of total energy*, J. of Comput. Phys. vol: 146, pp: 227-262 (1998).
- [3] P.-H. Maire, *A high-order cell-centered Lagrangian scheme for compressible fluid flows in two-dimensional cylindrical geometry*, J. of Comput. Phys. vol: 228, pp: 6882-6915 (2009).
- [4] W. F. Noh, *Errors for calculations of strong shocks using an artificial viscosity and an artificial heat flux*, J. of Comput. Phys. vol: 72, pp: 78-120 (1987).